Exact stability for Turán's theorem

Dániel Korándi

University of Oxford

May 7, 2020

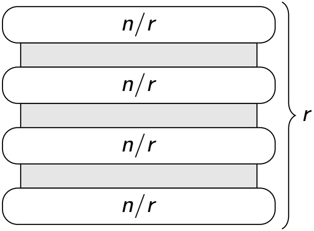
joint work with Alexander Roberts and Alex Scott

Let G be a K_{r+1}-free graph on n vertices.

Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

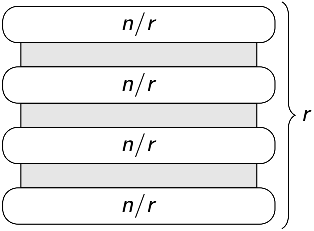


Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

$T_r(n)$ the unique extremal graph.



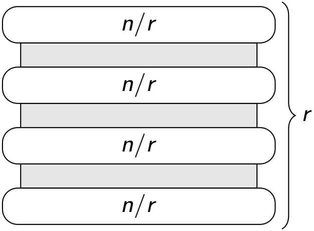
Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

$T_r(n)$ the unique extremal graph.

$t_r(n) = e(T_r(n)) \approx \frac{r-1}{2r} \cdot n^2$



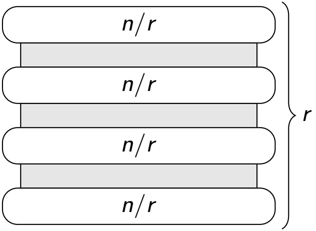
Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

$T_r(n)$ the unique extremal graph.

$t_r(n) = e(T_r(n)) \approx \frac{r-1}{2r} \cdot n^2$



Stability

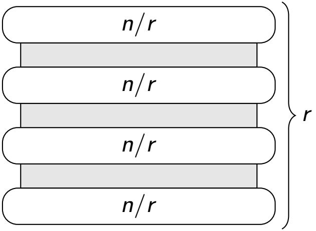
Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

$T_r(n)$ the unique extremal graph.

$t_r(n) = e(T_r(n)) \approx \frac{r-1}{2r} \cdot n^2$



Stability

close in size

$\Downarrow$

close in structure

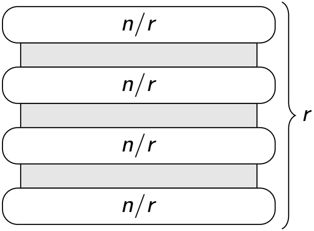
Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

$T_r(n)$ the unique extremal graph.

$t_r(n) = e(T_r(n)) \approx \frac{r-1}{2r} \cdot n^2$



Stability

close in size — # edges

$\Downarrow$

close in structure

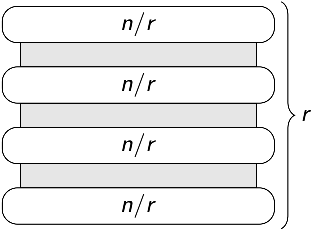
Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

$T_r(n)$ the unique extremal graph.

$t_r(n) = e(T_r(n)) \approx \frac{r-1}{2r} \cdot n^2$



Stability

close in size — $\#$ edges

$\Downarrow$

close in structure — distance from $T_r(n)$

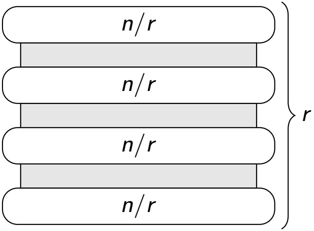
Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

$T_r(n)$ the unique extremal graph.

$$t_r(n) = e(T_r(n)) \approx \frac{r-1}{2r} \cdot n^2$$



Stability

close in size — # edges

$\Downarrow$

close in structure < distance from $T_r(n)$; distance from being r-partite

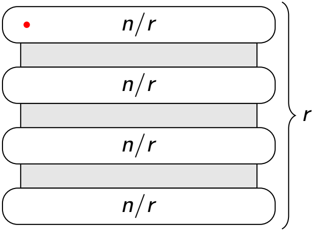
Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

$T_r(n)$ the unique extremal graph.

$$t_r(n) = e(T_r(n)) \approx \frac{r-1}{2r} \cdot n^2$$



Stability

close in size — # edges

⇓

close in structure
- distance from $T_r(n)$
- distance from being r-partite

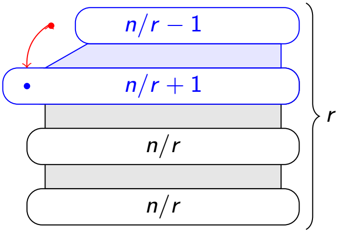
Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

$T_r(n)$ the unique extremal graph.

$t_r(n) = e(T_r(n)) \approx \frac{r-1}{2r} \cdot n^2$



Stability

close in size — # edges

$\Downarrow$

close in structure
- distance from $T_r(n)$
- distance from being r-partite

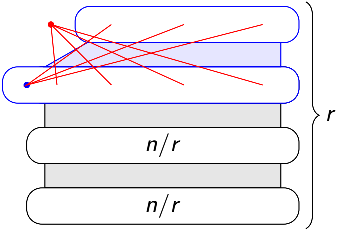
Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

$T_r(n)$ the unique extremal graph.

$t_r(n) = e(T_r(n)) \approx \frac{r-1}{2r} \cdot n^2$



Stability

close in size — # edges

$\Downarrow$

close in structure
- distance from $T_r(n)$
- distance from being r-partite

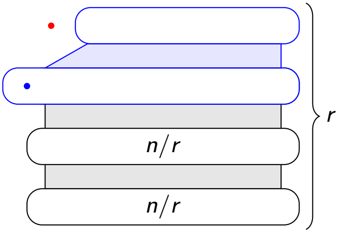
Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

$T_r(n)$ the unique extremal graph.

$$t_r(n) = e(T_r(n)) \approx \frac{r-1}{2r} \cdot n^2$$



Stability

close in size — # edges

⇓

close in structure: distance from $T_r(n)$; distance from being r-partite

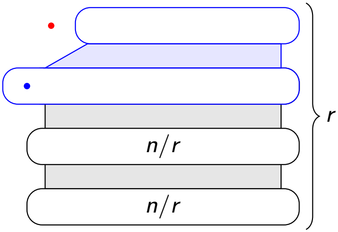
Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

$T_r(n)$ the unique extremal graph.

$t_r(n) = e(T_r(n)) \approx \frac{r-1}{2r} \cdot n^2$



Stability

close in size — # edges $= e(G)$

$\Downarrow$

close in structure — distance from $T_r(n)$

distance from being r-partite $= D_r(G)$

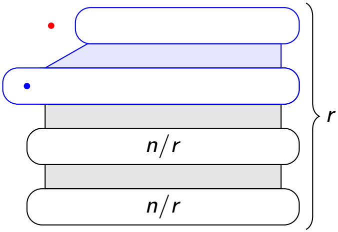
Let G be a K_{r+1}-free graph on n vertices.

Turán's theorem (1941)

The number of edges in G is largest when G is the Turán graph $T_r(n)$.

$T_r(n)$ the unique extremal graph.

$$t_r(n) = e(T_r(n)) \approx \frac{r-1}{2r} \cdot n^2$$



Stability

close in size — # edges $= e(G)$

$\Downarrow$

close in structure < distance from $T_r(n)$

distance from being r-partite $= D_r(G)$

Want to bound $D_r(G)$ when $e(G) \geq t_r(n) - t$

Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$

Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$

Erdős-Simonovits Stability Theorem (1966):

- $e(G) = t_r(n) - o(n^2) \quad \Rightarrow \quad D_r(G) = o(n^2)$.

Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$

Erdős-Simonovits Stability Theorem (1966):

- $e(G) = t_r(n) - o(n^2) \quad \Rightarrow \quad D_r(G) = o(n^2)$.

Füredi (2006-2015):

- $e(G) = t_r(n) - t \quad \Rightarrow \quad D_r(G) \leq t$.

Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$

Erdős-Simonovits Stability Theorem (1966):

- $e(G) = t_r(n) - o(n^2) \quad \Rightarrow \quad D_r(G) = o(n^2)$.

Füredi (2006-2015):

- $e(G) = t_r(n) - t \quad \Rightarrow \quad D_r(G) \leq t$.

Roberts-Scott (2018):

- $t \leq \delta n^2 \quad \Rightarrow \quad D_r(G) \leq O(\frac{t^{3/2}}{n})$.

Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$

Erdős-Simonovits Stability Theorem (1966):

- $e(G) = t_r(n) - o(n^2) \quad \Rightarrow \quad D_r(G) = o(n^2)$.

Füredi (2006-2015):

- $e(G) = t_r(n) - t \quad \Rightarrow \quad D_r(G) \leq t$.

Roberts-Scott (2018):

- $t \leq \delta n^2 \quad \Rightarrow \quad D_r(G) \leq O(\frac{t^{3/2}}{n})$.
- $t = \delta n^2 \quad \Rightarrow \quad \max D_r(G) = \Theta(\delta^{3/2} n^2)$.

Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$

Erdős-Simonovits Stability Theorem (1966):

- $e(G) = t_r(n) - o(n^2) \quad \Rightarrow \quad D_r(G) = o(n^2)$.

Füredi (2006-2015):

- $e(G) = t_r(n) - t \quad \Rightarrow \quad D_r(G) \leq t$.

Roberts-Scott (2018):

- $t \leq \delta n^2 \quad \Rightarrow \quad D_r(G) \leq O(\frac{t^{3/2}}{n})$.
- $t = \delta n^2 \quad \Rightarrow \quad \max D_r(G) = \Theta(\delta^{3/2} n^2)$.

Balogh-Clemen-Lavrov-Lidický-Pfender (2019+):

- $t = \delta n^2, \delta = o(1) \quad \Rightarrow \quad \max D_r(G) = \left(\frac{2r}{3\sqrt{3}} + o(1)\right) \delta^{3/2} n^2$.

Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$

Exact results

Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$

Exact results

$\mathbf{D_r(G) = 0}$:

Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$

Exact results

$\mathbf{D_r(G) = 0}$:

Brouwer (1981)

If G is K_{r+1}-free with $e(G) \geq t_r(n) - \lfloor \frac{n}{r} \rfloor + 2$, then G is r-partite.

Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$

Exact results

$\mathbf{D_r(G) = 0}$:

Brouwer (1981)

If G is K_{r+1}-free with $e(G) \geq t_r(n) - \lfloor \frac{n}{r} \rfloor + 2$, then G is r-partite.

$\mathbf{r = 2}$:

Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.

How many edges to be deleted to make G r-partite? $= D_r(G)$

Exact results

$\mathbf{D_r(G) = 0}$:

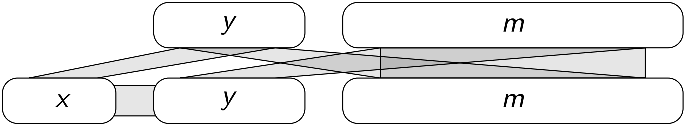
Brouwer (1981)

If G is K_{r+1}-free with $e(G) \geq t_r(n) - \lfloor \frac{n}{r} \rfloor + 2$, then G is r-partite.

$\mathbf{r = 2}$:

Erdős-Győri-Simonovits (1992)

The "worst" K_3-free graphs are C_5-blowups.



Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$

Exact results

$\mathbf{D_r(G) = 0}$:

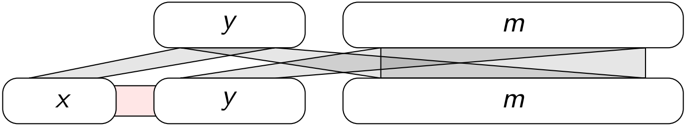
Brouwer (1981)

If G is K_{r+1}-free with $e(G) \geq t_r(n) - \lfloor \frac{n}{r} \rfloor + 2$, then G is r-partite.

$\mathbf{r = 2}$:

Erdős-Győri-Simonovits (1992)

The "worst" K_3-free graphs are C_5-blowups.



Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.

How many edges to be deleted to make G r-partite? $= D_r(G)$

Exact results

$\mathbf{D_r(G) = 0}$:

Brouwer (1981)

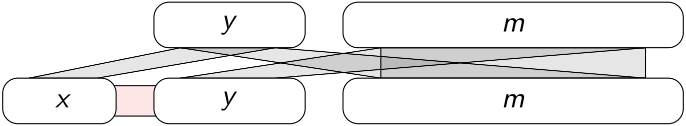
If G is K_{r+1}-free with $e(G) \geq t_r(n) - \lfloor \frac{n}{r} \rfloor + 2$, then G is r-partite.

$\mathbf{r = 2}$:

Erdős-Győri-Simonovits (1992)

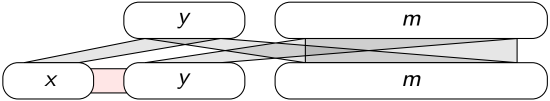
The "worst" K_3-free graphs are C_5-blowups.

For every K_3-free graph G with $e(G) \geq \frac{n^2}{5}$, there is a C_5-blowup H s.t. $e(H) \geq e(G)$ and $D_2(H) \geq D_2(G)$.



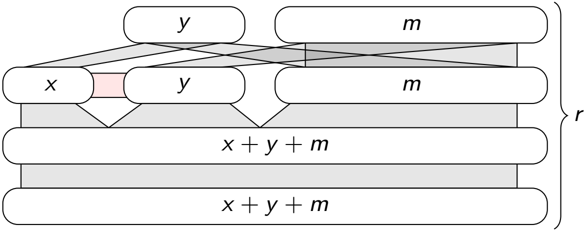
Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$



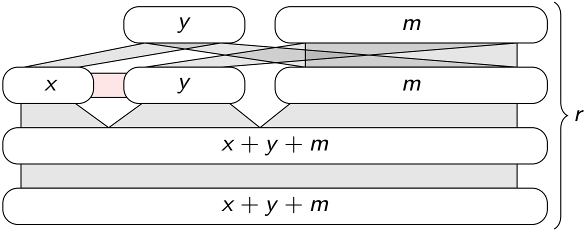
Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$



Problem

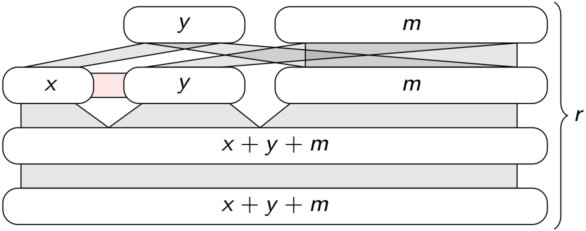
G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$



pentagonal Turán graph

Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$



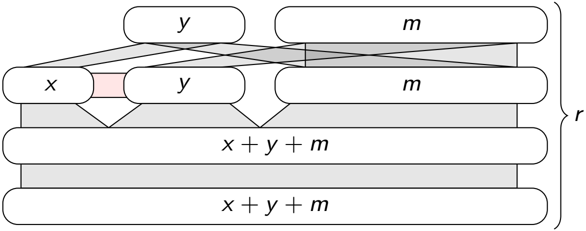
pentagonal Turán graph

Conjecture (Balogh-Clemen-Lavrov-Lidický-Pfender)

For every K_{r+1}-free G with $e(G) \geq t_r(n) - \delta n^2$, there is a pentagonal Turán graph H s.t. $e(H) \geq e(G)$ and $D_r(H) \geq D_r(G)$.

Problem

G is a K_{r+1}-free graph with $e(G) = t_r(n) - t$.
How many edges to be deleted to make G r-partite? $= D_r(G)$



pentagonal Turán graph

Conjecture (Balogh-Clemen-Lavrov-Lidický-Pfender)

For every K_{r+1}-free G with $e(G) \geq t_r(n) - \delta n^2$, there is a pentagonal Turán graph H s.t. $e(H) \geq e(G)$ and $D_r(H) \geq D_r(G)$.

Theorem (K-Roberts-Scott, 2020+)

The conjecture holds ($\delta = r^{-60}$ works).

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

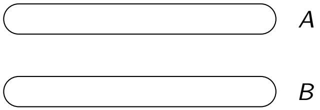
Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

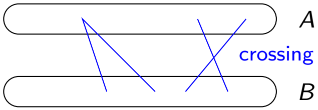
- Take max cut in G



Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

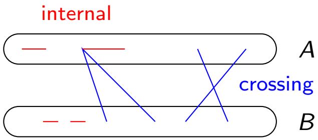
- Take max cut in G



Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

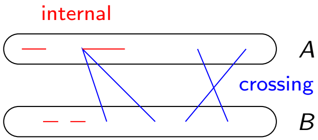
- Take max cut in G



Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

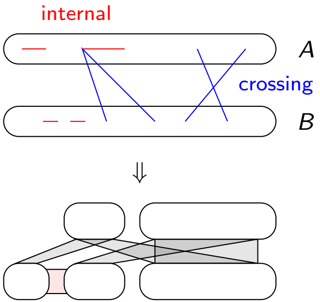
- Take max cut in G
- $D_2(G) \leq t := \#$ internal edges



Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

- Take max cut in G
- $D_2(G) \leq t := \#$ internal edges
- Construct H with $D_2(H) = t$

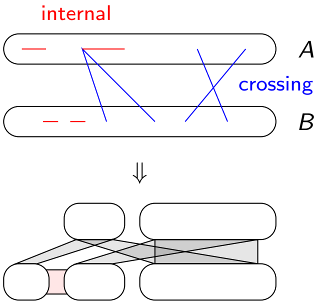


C_5-blowup H

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

- Take max cut in G
- $D_2(G) \leq t := \#$ internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \leq e(H)$



C_5-blowup H

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

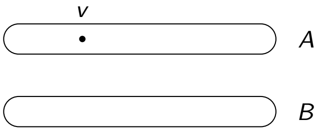
- Take max cut in G
- $D_2(G) \leq t := \#$ internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \leq e(H)$

To bound $e(G)$: 3 arguments

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

- Take max cut in G
- $D_2(G) \leq t := \#$ internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \leq e(H)$



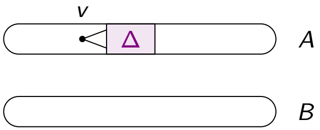
To bound $e(G)$: 3 arguments

1.

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

- Take max cut in G
- $D_2(G) \leq t := \#$ internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \leq e(H)$



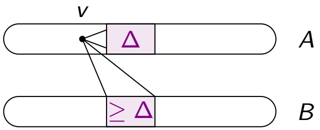
To bound $e(G)$: 3 arguments

1. **[neighbors of v]**

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

- Take max cut in G
- $D_2(G) \leq t := \#$ internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \leq e(H)$



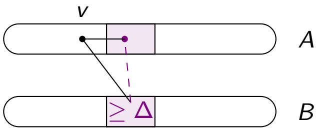
To bound $e(G)$: 3 arguments

1. **[neighbors of v]**

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

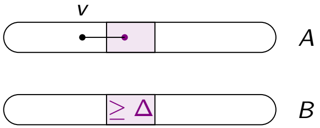
- Take max cut in G
- $D_2(G) \leq t := \#$ internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \leq e(H)$



To bound $e(G)$: 3 arguments

1. **[neighbors of v]**

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)



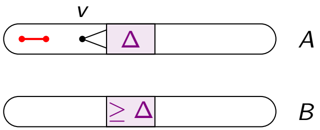
G is a K_3-free graph

- Take max cut in G
- $D_2(G) \leq t := \#$ internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \leq e(H)$

To bound $e(G)$: 3 arguments

1. **[neighbors of v]**
   Each internal neighbor of v touches $\leq |B| - \Delta$ crossing edges

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)



G is a K_3-free graph

- Take max cut in G
- $D_2(G) \leq t := \#$ internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \leq e(H)$

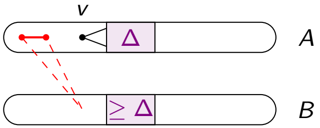
To bound $e(G)$: 3 arguments

1. **[neighbors of v]**
   Each internal neighbor of v touches $\leq |B| - \Delta$ crossing edges
2. **[internal edge]**

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

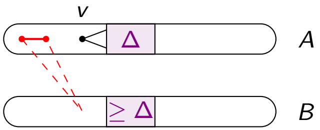
- Take max cut in G
- $D_2(G) \leq t :=$ # internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \leq e(H)$



To bound $e(G)$: 3 arguments

1. **[neighbors of v]**
   Each internal neighbor of v touches $\leq |B| - \Delta$ crossing edges
2. **[internal edge]**

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)



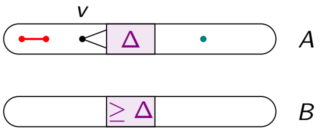
G is a K_3-free graph

- Take max cut in G
- $D_2(G) \leq t := \#$ internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \leq e(H)$

To bound $e(G)$: 3 arguments

1. **[neighbors of v]**
   Each internal neighbor of v touches $\leq |B| - \Delta$ crossing edges
2. **[internal edge]**
   Every edge in A touches $\leq |B|$ crossing edges

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)



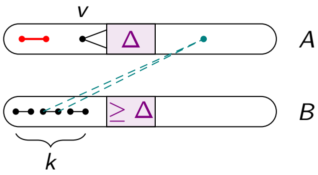
G is a K_3-free graph

- Take max cut in G
- $D_2(G) \leq t :=$ # internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \leq e(H)$

To bound $e(G)$: 3 arguments

1. **[neighbors of v]**
   Each internal neighbor of v touches $\leq |B| - \Delta$ crossing edges
2. **[internal edge]**
   Every edge in A touches $\leq |B|$ crossing edges
3. **[rest]**

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)



G is a K_3-free graph

- Take max cut in G
- $D_2(G) \leq t := \#$ internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \leq e(H)$

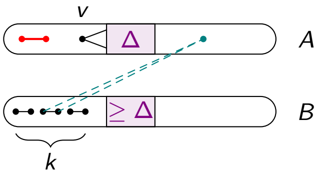
To bound $e(G)$: 3 arguments

1. **[neighbors of v]**
   Each internal neighbor of v touches $\leq |B| - \Delta$ crossing edges
2. **[internal edge]**
   Every edge in A touches $\leq |B|$ crossing edges
3. **[rest]**

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

- Take max cut in G
- $D_2(G) \le t := \#$ internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \le e(H)$



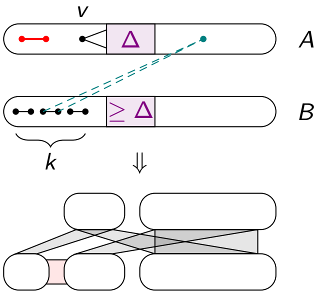
To bound $e(G)$: 3 arguments

1. **[neighbors of v]**
   Each internal neighbor of v touches $\le |B| - \Delta$ crossing edges
2. **[internal edge]**
   Every edge in A touches $\le |B|$ crossing edges
3. **[rest]**
   Every other vertex in A touches $\le |B| - k$ crossing edges

Proof sketch for $r = 2$ (Erdős-Győri-Simonovits)

G is a K_3-free graph

- Take max cut in G
- $D_2(G) \leq t := \#$ internal edges
- Construct H with $D_2(H) = t$
- Show that $e(G) \leq e(H)$



C_5-blowup H

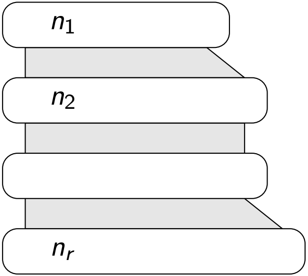
To bound $e(G)$: 3 arguments

1. **[neighbors of v]**
   Each internal neighbor of v touches $\leq |B| - \Delta$ crossing edges
2. **[internal edge]**
   Every edge in A touches $\leq |B|$ crossing edges
3. **[rest]**
   Every other vertex in A touches $\leq |B| - k$ crossing edges

A useful lemma

A useful lemma

Let $L = K_{n_1,\ldots,n_r}$ with $n_1 \leq \cdots \leq n_r$.

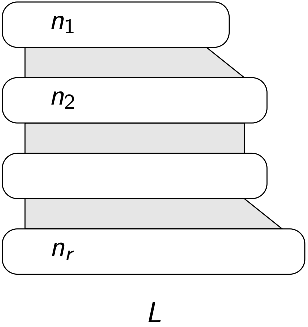


L

A useful lemma

Let $L = K_{n_1,\ldots,n_r}$ with $n_1 \leq \cdots \leq n_r$.

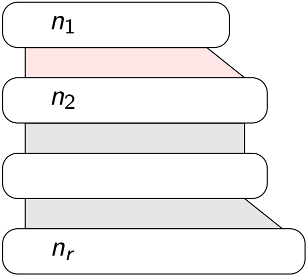
$\#$ edges in K_r-free subgraph?



A useful lemma

Let $L = K_{n_1,\ldots,n_r}$ with $n_1 \le \cdots \le n_r$.

edges in K_r-free subgraph?



L

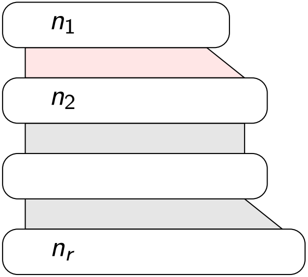
A useful lemma

Let $L = K_{n_1,\dots,n_r}$ with $n_1 \le \cdots \le n_r$.

$\#$ edges in K_r-free subgraph?

Lemma (folklore)

Every K_r-free subgraph of L misses at least $n_1 n_2$ edges.



L

A useful lemma

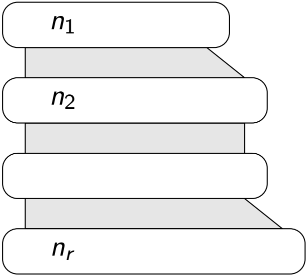
Let $L = K_{n_1,\dots,n_r}$ with $n_1 \le \cdots \le n_r$.

$\#$ edges in K_r-free subgraph?

Lemma (folklore)

Every K_r-free subgraph of L misses at least $n_1 n_2$ edges.

Proof.



L

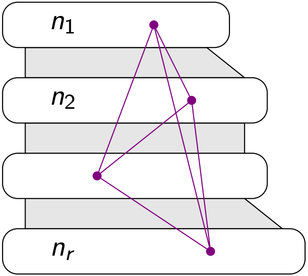
A useful lemma

Let $L = K_{n_1,\ldots,n_r}$ with $n_1 \leq \cdots \leq n_r$.

$\#$ edges in K_r-free subgraph?

Lemma (folklore)

Every K_r-free subgraph of L misses at least $n_1 n_2$ edges.



L

Proof.

- L contains $n_1 n_2 \cdots n_r$ copies of K_r

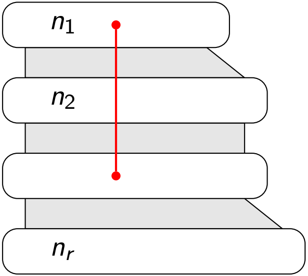
A useful lemma

Let $L = K_{n_1,\ldots,n_r}$ with $n_1 \le \cdots \le n_r$.

$\#$ edges in K_r-free subgraph?

Lemma (folklore)

Every K_r-free subgraph of L misses at least $n_1 n_2$ edges.



L

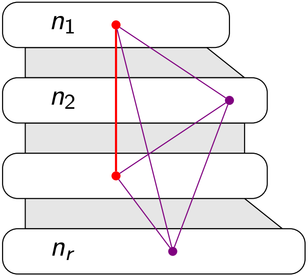
Proof.

- L contains $n_1 n_2 \cdots n_r$ copies of K_r

A useful lemma

Let $L = K_{n_1,\ldots,n_r}$ with $n_1 \le \cdots \le n_r$.

edges in K_r-free subgraph?



L

Lemma (folklore)

Every K_r-free subgraph of L misses at least $n_1 n_2$ edges.

Proof.

- L contains $n_1 n_2 \cdots n_r$ copies of K_r
- Each edge of L is contained in $\le n_3 \cdots n_r$ of them

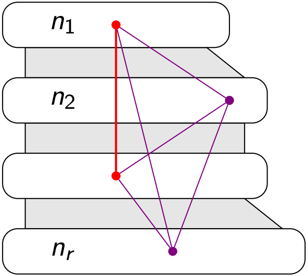
A useful lemma

Let $L = K_{n_1,\ldots,n_r}$ with $n_1 \leq \cdots \leq n_r$.

$\#$ edges in K_r-free subgraph?

Lemma (folklore)

Every K_r-free subgraph of L misses at least $n_1 n_2$ edges.



L

Proof.

- L contains $n_1 n_2 \cdots n_r$ copies of K_r
- Each edge of L is contained in $\leq n_3 \cdots n_r$ of them
- So at least $n_1 n_2$ edges need to be deleted from L to obtain a K_r-free subgraph

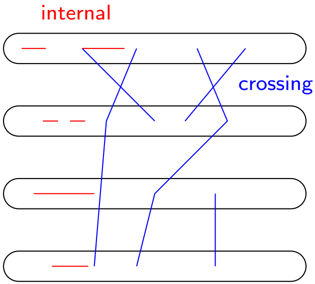
Proof sketch for general r

G is a K_{r+1}-free graph

Proof sketch for general r

G is a K_{r+1}-free graph

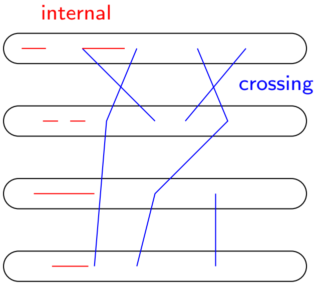
- Take max r-partition in G



Proof sketch for general r

G is a K_{r+1}-free graph

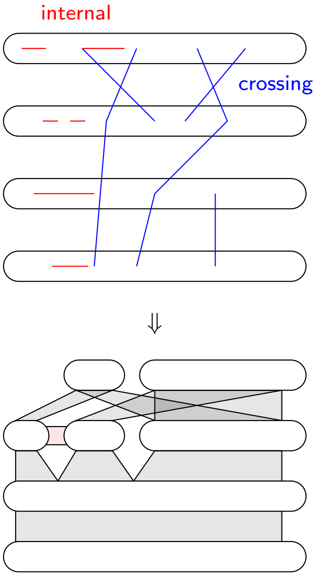
- Take max r-partition in G
- $D_r(G) \leq t := \#$ internal edges



Proof sketch for general r

G is a K_{r+1}-free graph

- Take max r-partition in G
- $D_r(G) \leq t := \#$ internal edges
- Construct H with $D_r(H) = t$

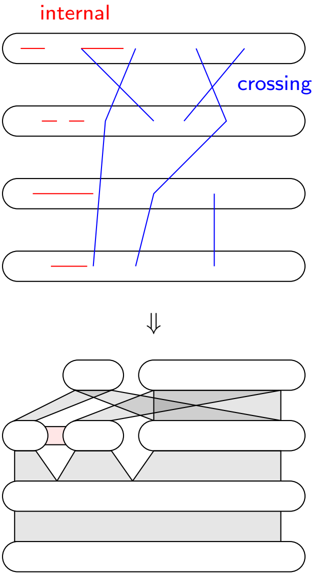


pentagonal Turán graph H

Proof sketch for general r

G is a K_{r+1}-free graph

- Take max r-partition in G
- $D_r(G) \leq t := \#$ internal edges
- Construct H with $D_r(H) = t$
- Show that $e(G) \leq e(H)$

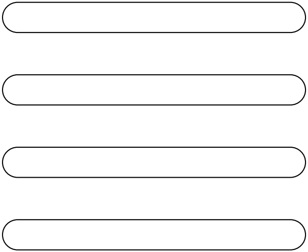


pentagonal Turán graph H

Proof sketch for general r

G is a K_{r+1}-free graph

- Take max r-partition in G
- $D_r(G) \leq t := \#$ internal edges
- Construct H with $D_r(H) = t$
- Show that $e(G) \leq e(H)$



To bound $e(G)$:

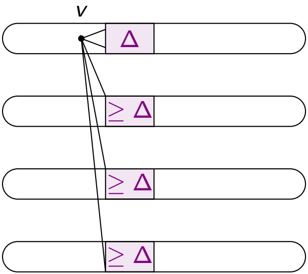
Proof sketch for general r

G is a K_{r+1}-free graph

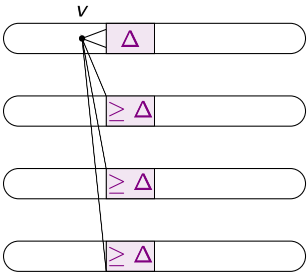
- Take max r-partition in G
- $D_r(G) \leq t :=$ # internal edges
- Construct H with $D_r(H) = t$
- Show that $e(G) \leq e(H)$

To bound $e(G)$:

1. **[neighbors of v]**



Proof sketch for general r



G is a K_{r+1}-free graph

- Take max r-partition in G
- $D_r(G) \leq t := \#$ internal edges
- Construct H with $D_r(H) = t$
- Show that $e(G) \leq e(H)$

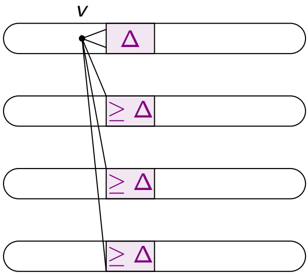
To bound $e(G)$:

1. **[neighbors of v]**
   crossing edges: K_r-free r-partite

Proof sketch for general r

G is a K_{r+1}-free graph

- Take max r-partition in G
- $D_r(G) \leq t := \#$ internal edges
- Construct H with $D_r(H) = t$
- Show that $e(G) \leq e(H)$



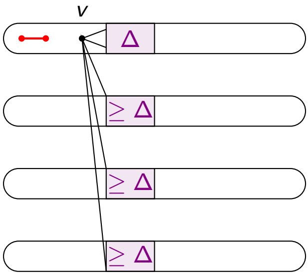
To bound $e(G)$:

1. **[neighbors of v]**
   crossing edges: K_r-free r-partite
   Lemma $\Rightarrow \Delta^2$ of them missing

Proof sketch for general r

G is a K_{r+1}-free graph

- Take max r-partition in G
- $D_r(G) \leq t := \#$ internal edges
- Construct H with $D_r(H) = t$
- Show that $e(G) \leq e(H)$



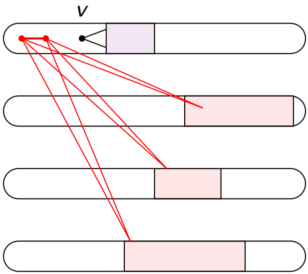
To bound $e(G)$:

1. **[neighbors of v]**
   crossing edges: K_r-free r-partite
   Lemma $\Rightarrow \Delta^2$ of them missing
2. **[internal edge]**

Proof sketch for general r

G is a K_{r+1}-free graph

- Take max r-partition in G
- $D_r(G) \leq t := \#$ internal edges
- Construct H with $D_r(H) = t$
- Show that $e(G) \leq e(H)$



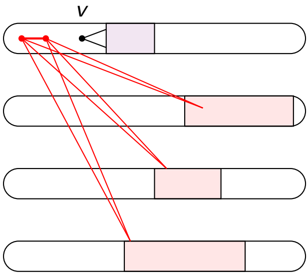
To bound $e(G)$:

1. **[neighbors of v]**
   crossing edges: K_r-free r-partite
   Lemma $\Rightarrow \Delta^2$ of them missing
2. **[internal edge]**

Proof sketch for general r

G is a K_{r+1}-free graph

- Take max r-partition in G
- $D_r(G) \leq t := \#$ internal edges
- Construct H with $D_r(H) = t$
- Show that $e(G) \leq e(H)$



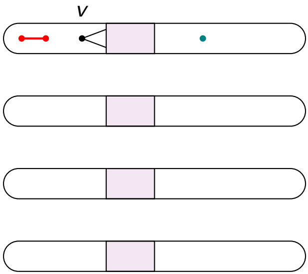
To bound $e(G)$:

1. **[neighbors of v]**
   crossing edges: K_r-free r-partite
   Lemma $\Rightarrow \Delta^2$ of them missing
2. **[internal edge]**
   - small common degree, **OR**
   - many missing edges

Proof sketch for general r

G is a K_{r+1}-free graph

- Take max r-partition in G
- $D_r(G) \leq t := \#$ internal edges
- Construct H with $D_r(H) = t$
- Show that $e(G) \leq e(H)$



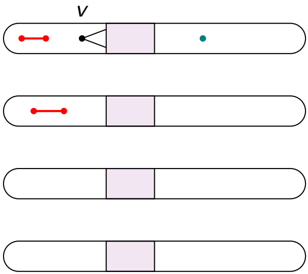
To bound $e(G)$:

1. **[neighbors of v]**
   crossing edges: K_r-free r-partite
   Lemma $\Rightarrow \Delta^2$ of them missing
2. **[internal edge]**
   - small common degree, **OR**
   - many missing edges
3. **[rest]**

Proof sketch for general r

G is a K_{r+1}-free graph

- Take max r-partition in G
- $D_r(G) \leq t :=$ # internal edges
- Construct H with $D_r(H) = t$
- Show that $e(G) \leq e(H)$

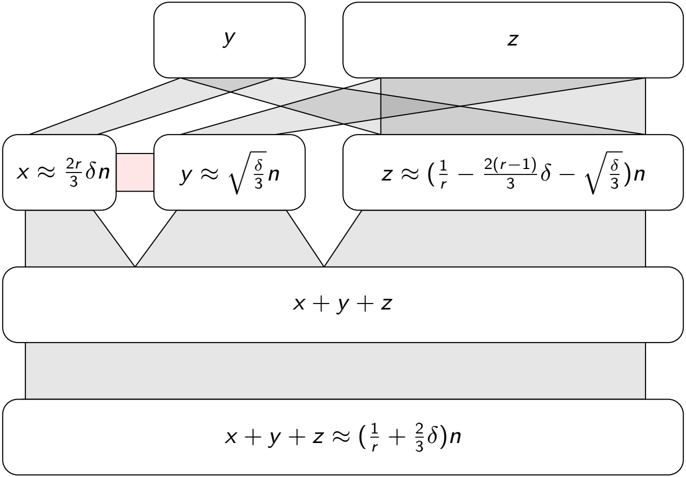


To bound $e(G)$:

1. **[neighbors of v]**
   crossing edges: K_r-free r-partite
   Lemma $\Rightarrow \Delta^2$ of them missing
2. **[internal edge]**
   - small common degree, **OR**
   - many missing edges
3. ~~**[rest]**~~ $\rightarrow$ use 2. in other parts

Theorem

Among n-vertex K_{r+1}-free graphs G with $t_r(n) - \delta n^2$ edges, $D_r(G)$ is maximized by:



Open problems

Open problems

1. Find exact stability for the extremal problem of other graphs.

Open problems

1. Find exact stability for the extremal problem of other graphs.

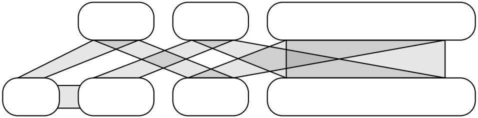
   Simonovits: $T_r(n)$ is extremal for $(r+1)$-chromatic H with critical edge.

Open problems

1. Find exact stability for the extremal problem of other graphs.

   Simonovits: $T_r(n)$ is extremal for $(r+1)$-chromatic H with critical edge.

   Conjecture (K-Roberts-Scott): The C_{2k-1}-free graphs G with largest $D_2(G)$ are C_{2k+1}-blowups.

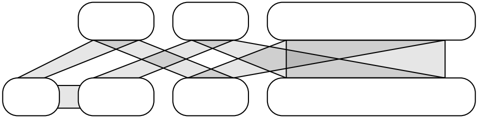


Open problems

1. Find exact stability for the extremal problem of other graphs.

   Simonovits: $T_r(n)$ is extremal for $(r+1)$-chromatic H with critical edge.

   Conjecture (K-Roberts-Scott): The C_{2k-1}-free graphs G with largest $D_2(G)$ are C_{2k+1}-blowups.

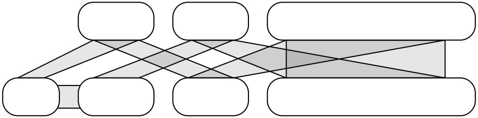
2. Conjecture (Erdős): $D_2(G) \leq \frac{n^2}{25}$ for every K_3-free G.

Open problems

1. Find exact stability for the extremal problem of other graphs.

   Simonovits: $T_r(n)$ is extremal for $(r+1)$-chromatic H with critical edge.

   Conjecture (K-Roberts-Scott): The C_{2k-1}-free graphs G with largest $D_2(G)$ are C_{2k+1}-blowups.

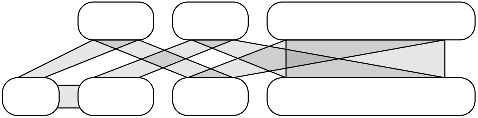
2. Conjecture (Erdős): $D_2(G) \leq \frac{n^2}{25}$ for every K_3-free G.
   - known for $e(G) \geq \frac{n^2}{5}$ (Erdős-Faudree-Pach-Spencer, 1988)

Open problems

1. Find exact stability for the extremal problem of other graphs.

   Simonovits: $T_r(n)$ is extremal for $(r+1)$-chromatic H with critical edge.

   Conjecture (K-Roberts-Scott): The C_{2k-1}-free graphs G with largest $D_2(G)$ are C_{2k+1}-blowups.

2. Conjecture (Erdős): $D_2(G) \le \frac{n^2}{25}$ for every K_3-free G.
   - known for $e(G) \ge \frac{n^2}{5}$ (Erdős-Faudree-Pach-Spencer, 1988)

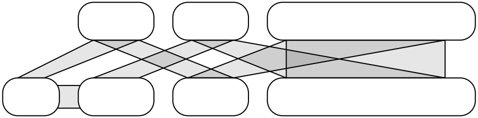
   Conjecture (Sudakov): $D_2(G) \le D_2(T_r(n))$ for every K_{r+1}-free G.

Open problems

1. Find exact stability for the extremal problem of other graphs.

   Simonovits: $T_r(n)$ is extremal for $(r+1)$-chromatic H with critical edge.

   Conjecture (K-Roberts-Scott): The C_{2k-1}-free graphs G with largest $D_2(G)$ are C_{2k+1}-blowups.



2. Conjecture (Erdős): $D_2(G) \leq \frac{n^2}{25}$ for every K_3-free G.
   - known for $e(G) \geq \frac{n^2}{5}$ (Erdős-Faudree-Pach-Spencer, 1988)

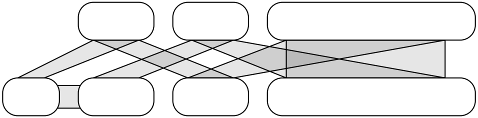
   Conjecture (Sudakov): $D_2(G) \leq D_2(T_r(n))$ for every K_{r+1}-free G.
   - known for $r = 3$ (Sudakov, 2007)

Open problems

1. Find exact stability for the extremal problem of other graphs.

   Simonovits: $T_r(n)$ is extremal for $(r+1)$-chromatic H with critical edge.

   Conjecture (K-Roberts-Scott): The C_{2k-1}-free graphs G with largest $D_2(G)$ are C_{2k+1}-blowups.



2. Conjecture (Erdős): $D_2(G) \leq \frac{n^2}{25}$ for every K_3-free G.
   - known for $e(G) \geq \frac{n^2}{5}$ (Erdős-Faudree-Pach-Spencer, 1988)

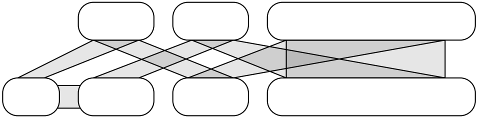
   Conjecture (Sudakov): $D_2(G) \leq D_2(T_r(n))$ for every K_{r+1}-free G.
   - known for $r = 3$ (Sudakov, 2007)
     and $r = 5$ (Hu, Lidický, Martins, Norin, Volec, 202x+)

Open problems

1. Find exact stability for the extremal problem of other graphs.

   Simonovits: $T_r(n)$ is extremal for $(r+1)$-chromatic H with critical edge.

   Conjecture (K-Roberts-Scott): The C_{2k-1}-free graphs G with largest $D_2(G)$ are C_{2k+1}-blowups.



2. Conjecture (Erdős): $D_2(G) \leq \frac{n^2}{25}$ for every K_3-free G.
   - known for $e(G) \geq \frac{n^2}{5}$ (Erdős-Faudree-Pach-Spencer, 1988)

   Conjecture (Sudakov): $D_2(G) \leq D_2(T_r(n))$ for every K_{r+1}-free G.
   - known for $r = 3$ (Sudakov, 2007)
     and $r = 5$ (Hu, Lidický, Martins, Norin, Volec, 202x+)

Thank you!