# Powers of paths in tournaments

Nemanja Draganić* François Dross† Jacob Fox‡ António Girão§
Frédéric Havet¶ Dániel Korándi‖** William Lochet†† David Munhá Correia*
Alex Scott‖ Benny Sudakov*

**Abstract**

In this short note we prove that every tournament contains the $k$-th power of a directed path of linear length. This improves upon recent results of Yuster and of Girão. We also give a complete solution for this problem when $k = 2$, showing that there is always a square of a directed path of length $\lceil 2n/3 \rceil - 1$, which is best possible.

## 1 Introduction

One of the main themes in extremal graph theory is the study of embedding long paths and cycles in graphs. Some of the classical examples include the Erdős–Gallai theorem [3] that every $n$-vertex graph with average degree $d$ contains a path of length $d$, and Dirac's theorem [2] that every graph with minimum degree $n/2$ contains a Hamilton cycle. A famous generalization of this, conjectured by Pósa and Seymour, and proved for large $n$ by Komlós, Sárközy and Szemerédi [5], asserts that if the minimum degree is at least $kn/(k+1)$, then the graph contains the $k$-th power of a Hamilton cycle.

In this note, we are interested in embedding directed graphs in a tournament. A tournament is an oriented complete graph. The $k$-th power of the directed path $\vec{P}_\ell = v_0 \dots v_\ell$ of length $\ell$ is the graph $\vec{P}_\ell^k$ on the same vertex set containing a directed edge $v_i v_j$ if and only if $i < j \leq i + k$. The $k$-th power of a directed cycle is defined analogously. An old result of Bollobás and Häggkvist [1] says that, for large $n$, every $n$-vertex tournament with all indegrees and outdegrees at least $(1/4 + \varepsilon)n$ contains the $k$-th power of a Hamilton cycle

*Department of Mathematics, ETH Zurich, Zurich, Switzerland. Emails: {nemanja.draganic, david.munhacanascorreia, benjamin.sudakov}@math.ethz.ch. Research supported in part by SNSF grant 200021_196965

†Institute of Informatics, University of Warsaw, Poland. Email: francois.dross@googlemail.com. Research supported in part by ERC grant No 714704.

‡Department of Mathematics, Stanford University, Stanford, CA, USA. Email: jacobfox@stanford.edu. Research supported by a Packard Fellowship and by NSF award DMS-185563.

§E-mail: tzgirao@gmail.com.

¶CNRS, Université Côte d'Azur, I3S, INRIA, Sophia Antipolis, France. Email: frederic.havet@inria.fr. Research supported in part by Agence Nationale de la Recherche under contract Digraphs ANR-19-CE48-0013-01.

‖Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, United Kingdom. Emails: {korandi, scott}@maths.ox.ac.uk.

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††Department of Informatics, University of Bergen, Norway. Email: william.lochet@uib.no. Research supported by ERC grant No 819416.

(the constant 1/4 is optimal). However, we cannot expect to find powers of directed cycles in general, as the transitive tournament contains no cycles at all.

What about powers of directed paths? A classical result, which appears in every graph theory book (see, e.g., [7]), says that every tournament contains a directed Hamilton path. On the other hand, Yuster [6] recently observed that some tournaments are quite far from containing the square of a Hamilton path. In particular, there is an $n$-vertex tournament that does not even contain the square of $\vec{P}_{2n/3}$, and more generally, for every $k \geq 2$, there are tournaments with $n$ vertices and no $k$-th power of a path with more than $nk/2^{k/2}$ vertices. In the other direction, Yuster proved that every tournament with $n$ vertices contains the square of a path of length $n^{0.295}$. This was improved very recently by Girão [4], who showed that for fixed $k$, every tournament on $n$ vertices contains the $k$-th power of a path of length $n^{1-o(1)}$. Both papers noted that no sublinear upper bound is known. Our main result shows that the maximum length is in fact linear in $n$.

**Theorem 1.** *For $n \geq 2$, every $n$-vertex tournament contains the $k$-th power of a directed path of length $n/2^{4k+6}k$.*

The proof of this theorem combines Kővári–Sós–Turán style arguments, used for the bipartite Turán problem, and median orderings of tournaments. A median ordering is a vertex ordering that maximizes the number of forward edges. Theorem 1 and Yuster's construction show that an optimal bound on the length has the form $n/2^{\Theta(k)}$. It would be interesting to find the exact value of the constant factor in the exponent. Optimizing our proof can yield a lower bound of $n/2^{ck+o(k)}$ with $c \approx 3.9$, but is unlikely to give the correct bound.

We also improve the exponential constant in the upper bound from 1/2 to 1.

**Theorem 2.** *Let $k \geq 5$ and $n \geq k(k+1)2^k$. There is an $n$-vertex tournament that does not contain the $k$-th power of a directed path of length $k(k+1)n/2^k$.*

Note that this theorem also holds trivially for $k \leq 4$, when $k(k+1)n/2^k > n$.

Finally, we can solve the problem completely in the special case of $k = 2$. Once again, the proof uses certain properties of median orderings.

**Theorem 3.** *For $n \geq 1$, every $n$-vertex tournament contains the square of a directed path of length $\ell = \lceil 2n/3 \rceil - 1$, but not necessarily of length $\ell + 1$.*

Theorems 1, 2 and 3 are proved in Sections 2, 3 and 4, respectively.

## 2 Lower bound

We will need the following Kővári–Sós–Turán style lemma.

**Lemma 4.** *Let $G$ be a directed graph with disjoint vertex subsets $A$ and $B$ with $|A| = 2k+1$, $|B| \geq 2^{4k+4}k$, and every vertex in $A$ has at least $(1 - \frac{1}{2k+1})|B|/2$ outneighbours in $B$. Then $A$ contains a subset $A'$ of size $k$ that has at least $(2k+1)2^{2k}$ common outneighbours in $B$.*

*Proof.* Suppose there is no such set $A'$. Then every $k$-subset of $A$ appears in the inneighbourhood of less than $(2k+1)2^{2k}$ vertices in $B$. So if $d^-(v)$ denotes the number of inneighbours a vertex $v \in B$ has in $A$, then we have

$$\binom{2k+1}{k} \cdot (2k+1)2^{2k} = \binom{|A|}{k} \cdot (2k+1)2^{2k} > \sum_{v \in B} \binom{d^-(v)}{k}. \tag{1}$$

On the other hand, $\sum_{v \in B} d^-(v) \geq |A|(1 - \frac{1}{2k+1})|B|/2 = k|B|$. By Jensen's inequality, $\sum_{v \in B} \binom{d^-(v)}{k} \geq |B| \cdot \binom{\sum_{v \in B} d^-(v)/|B|}{k} = |B| \geq 2^{4k+4}k$. This contradicts (1). □

One more ingredient we need for the proof of Theorem 1 is the folklore fact that every tournament on $2^m$ vertices contains a transitive subtournament of size $m+1$. This is easily seen by taking a vertex of outdegree at least $2^{m-1}$ as the first vertex of the subtournament, and then recursing on the outneighbourhood.

*Proof of Theorem 1.* Order the vertices as $0, 1, \ldots, n-1$ to maximize the number of forward edges, i.e., the number of edges $ij$ such that $i < j$. As was mentioned in the introduction, we will refer to such a sequence as a *median ordering* of the vertices. We denote an "interval" of vertices with respect to this ordering by $[i, j) = \{i, \ldots, j-1\}$, where $0 \leq i < j \leq n$.

We will embed $\vec{P}_\ell^k$ inductively using the following claim.

**Claim.** Let $t = 2^{4k+4}k$ and $t \leq i \leq n - (2k+1)t$. For every subset $A^* \subseteq [i-t, i)$ of size $2^{2k}$, there is an index $i + t \leq j \leq i + (2k+1)t$ and a set $A' \subseteq A^*$ of size $k$ such that $A'$ induces a transitive tournament and its vertices have at least $2^{2k}$ common outneighbours in $[j-t, j)$.

*Proof.* There is a subset $A \subseteq A^*$ of size $2k+1$ that induces a transitive tournament. Let $B = [i, i + (2k+1)t)$. Then every vertex $v \in A$ has at least $kt = \left(1 - \frac{1}{2k+1}\right)|B|/2$ outneighbours in $B$. Indeed, otherwise $v$ would have more than $(k+1)t$ inneighbours in the interval $B$, so moving $v$ to the end of this interval would increase the number of forward edges in the ordering, contradicting our choice of the vertex ordering.

We can thus apply Lemma 4 to find a $k$-subset $A' \subseteq A$ with least $(2k+1)2^{2k}$ common outneighbours in $B$. Partition $B$ into $2k+1$ intervals of size $t$, and we can choose $j$ accordingly so that $A'$ has at least $2^{2k}$ common outneighbors in the interval $[j-t, j)$. □

The theorem trivially holds for $n < 2^{2k}$, so assume $n \geq 2^{2k}$. Let $i_0 = 2^{2k}$ and $A_0 = [0, 2^{2k})$, and apply the Claim with $i = i_0$ and $A^* = A_0$. We get a set $A' \subset A_0$ of size $k$ that induces a transitive tournament, i.e., the $k$-th power of some path $v_0 \ldots v_{k-1}$. Moreover, this $A'$ has at least $2^{2k}$ common outneighbours in some interval $[j-t, j)$ with $i_0 + t \leq j \leq i_0 + (2k+1)t$. Let us define $i_1 = j$, and choose $A_1$ to be any $2^{2k}$ of the common outneighbours.

At step $s$, we apply the Claim again with $i = i_s$ and $A^* = A_s$ to find the $k$-th power of some path $v_{sk} \ldots v_{(s+1)k-1}$ in $A_s$ with $2^{2k}$ common outneighbours in some $[i_{s+1} - t, i_{s+1})$ with $i_s + t \leq i_{s+1} \leq i_s + (2k+1)t$, and repeat this process until some step $\ell$ with $i_\ell > n - (2k+1)t$. Note that intervals $[i_s - t, i_s)$ and $[i_{s+1} - t, i_{s+1})$ are always disjoint. Finally, $A_\ell$ must also contain a transitive tournament of size $2k+1$. Call these vertices $v_{\ell k}, \ldots, v_{(\ell+2)k}$. Observe that $n - (2k+1)t < i_\ell \leq 2^{2k} + \ell(2k+1)t$, so $n < (\ell+2)(2k+1)t$.

Then $v_0 \dots v_{(\ell+2)k}$ is a directed path of length $(\ell+2)k \geq kn/(2k+1)t \geq n/(2^{4k+6}k)$ whose $k$-th power is contained in the tournament. In fact, we proved a bit more: the tournament contains all edges of the form $v_a v_b$ with $a < b$ and $\lfloor a/k \rfloor + 1 \geq \lfloor b/k \rfloor$. □

# 3 Upper bound

Let $\ell_k(n)$ denote the smallest integer $\ell$ such that there is an $n$-vertex tournament that does not contain $\vec{P}_\ell^k$, or in other words, the largest integer such that every $n$-vertex tournament contains the $k$-th power of a directed path on $\ell$ vertices.

To prove Theorem 2, we first note that $\ell_k(n)$ is subadditive.

**Lemma 5.** *For any $k, n, m \geq 1$, we have $\ell_k(n+m) \leq \ell_k(n) + \ell_k(m)$.*

*Proof.* Let $T_1$ and $T_2$ be extremal tournaments on $n$ and $m$ vertices, respectively, not containing the $k$-th power of any directed path of length $\ell_k(n)$ and $\ell_k(m)$. Let $T$ be the tournament on $n+m$ vertices, obtained from the disjoint union of $T_1$ and $T_2$ by adding all remaining edges directed from $T_1$ to $T_2$. Then any $k$-th power of a path in $T$ must be the concatenation of the $k$-th power of a path in $T_1$ and the $k$-th power of a path in $T_2$, and hence it must have length at most $(\ell_k(n)-1) + (\ell_k(m)-1) + 1 < \ell_k(n) + \ell_k(m)$. □

Our improved upper bound is based on the following construction.

**Lemma 6.** *For every $k \geq 5$, we have $\ell_k(2^{k-1}) < \frac{k(k+1)}{2}$.*

*Proof.* Let $n = 2^{k-1}$ and $\ell = \frac{k(k+1)}{2}$, and note that $\vec{P}_{\ell-1}^k$ has $k\ell - \ell$ edges.

Let $T$ be a random $n$-vertex tournament obtained by orienting the edges of $K_n$ independently and uniformly at random. The probability that a fixed sequence of $\ell$ vertices $v_0 \dots v_{\ell-1}$ forms a copy of $\vec{P}_{\ell-1}^k$ is $2^{-(k-1)\ell}$. There are $\binom{n}{\ell} \cdot \ell!$ such sequences, so the probability that $T$ contains the $k$-th power of a path of length $\ell-1$ is at most $\binom{n}{\ell} \cdot \ell! \cdot 2^{-(k-1)\ell} < n^\ell \cdot 2^{-(k-1)\ell} = 1$. So with positive probability $T$ does not contain $\vec{P}_{\ell-1}^k$, therefore $\ell_k(2^{k-1}) \leq \ell - 1$. □

Combining Lemmas 5 and 6 and using the monotonicity of $\ell_k(n)$, we get

$$\ell_k(n) \leq \left\lceil \frac{n}{2^{k-1}} \right\rceil \cdot \ell_k(2^{k-1}) \leq \left( \frac{n}{2^{k-1}} + 1 \right) \left( \frac{k(k+1)}{2} - 1 \right) \leq \frac{k(k+1)n}{2^k}$$

for $n \geq k(k+1)2^k$, establishing Theorem 2.

# 4 The square of a path

*Proof of Theorem 3.* Recall that $\ell_2(n)$ is the largest integer such that every $n$-vertex tournament contains the square of a path on $\ell$ vertices. Proving Theorem 3 is therefore equivalent to showing $\ell_2(n) = \lceil 2n/3 \rceil$ for every $n \geq 1$.

It is easy to check that $\ell_2(1) = 1$ and $\ell_2(2) = \ell_2(3) = 2$, so $\ell_2(n) \leq \lceil 2n/3 \rceil$ follows from Lemma 5 by induction, as $\ell_2(n) \leq \ell_2(n-3) + \ell_2(3) = \ell_2(n-3) + 2$ holds for every $n > 3$. For the lower bound we need to take a closer look at median orderings.

**Claim.** Every median ordering $x_1, \dots, x_n$ of a tournament has the following properties:

(a) All edges of the form $x_i x_{i+1}$ are in the tournament.

(b) If $x_i x_{i-2}$ is an edge of the tournament, then "rotating" $x_{i-2}x_{i-1}x_i$ gives two other median orderings $x_1, \dots, x_{i-3}, x_{i-1}, x_i, x_{i-2}, x_{i+1}, \dots, x_n$ and $x_1, \dots, x_{i-3}, x_i, x_{i-2}, x_{i-1}, x_{i+1}, \dots, x_n$.

(c) If $x_i x_{i-2}$ is an edge of the tournament, then each of $x_{i-2}, x_{i-1}, x_i$ is an inneighbour of $x_{i+1}$, and at most one of them is an outneighbour of $x_{i+2}$.

*Proof.* Property (a) holds, as otherwise we could swap $x_i$ and $x_{i+1}$ to get an ordering with more forward edges, contradicting our assumption. Property (b) holds because rotating $x_{i-2}x_{i-1}x_i$ has no effect on the number of forward edges.

These two properties together imply that each of $x_{i-2}, x_{i-1}, x_i$ is an inneighbour of $x_{i+1}$. Suppose, to the contrary of (c), that two of them are outneighbours of $x_{i+2}$. By rotating $x_{i-2}x_{i-1}x_i$ if needed, we may assume that these are $x_{i-1}$ and $x_i$. But then we can also rotate $x_i x_{i+1} x_{i+2}$ so that $x_{i+2}$ comes right after $x_{i-1}$ in a median ordering. This contradicts (a). $\square$

Let us now say that $i$ is a *bad index* in a median ordering $x_1, \dots, x_n$ if $x_i x_{i-2}$ is an edge, and at least one of $x_{i+2}x_i$ and $x_{i+2}x_{i-1}$ is also an edge.

**Lemma 7.** *Every tournament has a median ordering without any bad indices.*

*Proof.* Suppose this fails to hold for some tournament, and take a median ordering $x_1, \dots, x_n$ that minimizes the largest bad index $i$. As $i$ is a bad index, $x_i x_{i-2}$ is an edge, and $x_i$ or $x_{i-1}$ is an outneighbour of $x_{i+2}$. By (b), $x_{i-2}x_{i-1}x_i$ can be rotated so that $x_{i+2}x'_{i-2}$ is an edge in the new median ordering $x_1, \dots, x_{i-3}, x'_{i-2}, x'_{i-1}, x'_i, x_{i+1}, \dots, x_n$. Then neither $x_{i+2}x'_i$ nor $x_{i+2}x'_{i-1}$ is an edge, since by (c), only one of $x'_{i-2}, x'_{i-1}, x'_i$ is an outneighbour of $x_{i+2}$. Also by (c), $x'_{i-1}x_{i+1}$ and $x'_i x_{i+1}$ are edges, so both of $x_{i+1}$ and $x_{i+2}$ are outneighbours of $x'_{i-1}$ and $x'_i$. This means that none of $i, i+1, i+2$ is a bad index in this new ordering, and hence the largest bad index is smaller than $i$. This is a contradiction. $\square$

Now we are ready to prove $\ell_2(n) \geq \lceil 2n/3 \rceil$. Take an $n$-vertex tournament with median ordering $x_1, \dots, x_n$ as in Lemma 7, and let $I = \{i_1 < i_2 < \cdots < i_k\}$ be the set of indices $i$ such that $x_i x_{i-2}$ is not an edge (in particular, $i_1 = 1$ and $i_2 = 2$). We claim that $x_{i_1} \dots x_{i_k}$ is a directed path on $k \geq \lceil 2n/3 \rceil$ vertices whose square is contained in the tournament.

To see this, first observe that if the index $i+2$ is not in $I$, then both $i$ and $i+1$ are in $I$. Indeed, if $x_{i+2}x_i$ is an edge, then $x_{i+1}x_{i-1}$ cannot be one because of (c), and $x_i x_{i-2}$ cannot be one because $i$ is not a bad index. This immediately implies $k \geq \lceil 2n/3 \rceil$.

It remains to check that $x_{i_{j-2}}x_{i_j}$ and $x_{i_{j-1}}x_{i_j}$ are all edges in the tournament. By the above observation, we know that $i_j - 3 \leq i_{j-2} < i_{j-1} < i_j$. Here $x_{i_j-1}x_{i_j}$ is an edge by (a), and $x_{i_j-2}x_{i_j}$ is an edge by the definition of $I$. So the only case left is to show that $x_{i_{j-2}}x_{i_j}$ is an edge when $i_{j-2} = i_j - 3$.

In this case there is an index $i_j - 3 < i < i_j$ that is not in $I$, i.e., $x_i x_{i-2}$ is an edge in the tournament. But then if $i = i_j - 1$, then $x_{i_{j-2}}x_{i_j}$ is an edge because of (c), while otherwise $i = i_j - 2$, and $x_{i_{j-2}}x_{i_j}$ is an edge because $i$ is not a bad index. This concludes our proof. $\square$

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